CSCI 7000 Fall 2023: Permutations by exponential generating functions

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Due: Monday Oct 9, 2023

Recall: the *exponential generating function* of a sequence a_n is the power series $\sum_{n \geq 0} \frac{a_n}{n!} x^n$.

1. (a) A *fixed point* of a permutation π is an element such that $\pi(i) = i$. A *derangement* is a permutation with no fixed points. Let $d_n =$ the number of derangements of length n. Using generating function magic, find the exponential generating function for the sequence d_n. *Hint:* use the derivation in PS3 Question 5, but change the generating function for cycles to omit cycles of length 1.

   (b) Recall back to ordinary (not exponential) generating functions, what effect multiplying a generating function $A(z)$ by $1/(1-z)$ had. Use this together with part (a) of this question to give a simple summation formula for d_n.

   (c) What is $\lim_{n \to \infty} d_n/n!$ (that is, the probability that a randomly chosen permutation is a derangement)?

2. (a) **Skip (this problem turned out to either be much harder or much simpler than I intended, depending on how you try to do it; I suggest skipping it - the rest of Problem 2 is independent from this one)** Let ℓ_n be the average length of a cycle appearing in all permutations of a set of size n (that is: average over all cycles appearing in all permutations). Use generating functions to calculate ℓ_n.

   (b) Let $f_{n,k}$ be the number of permutations of length n with exactly k fixed points. Find a simple closed form for the "mixed

exponential–ordinary two-variable generating function"

$$\sum_{k,n\geq 0} f_{n,k} \frac{x^n y^k}{n!}.$$

Hint: Modify the exponential generating function for cycles by giving 1-cycles a weight of y, and other cycles a weight of 1.

(c) Let f_n be the average number of fixed points of a permutation of a set of size n (that is: average over all permutations). Use generating functions to calculate f_n. *Hint:* remember the $y\frac{d}{dy}$ operator.

(d) Let v_n be the variance of the number of fixed points of a permutation of a set of size n (that is: average over all permutations). Use generating functions to calculate v_n.

3. (a) Give a simple closed-form expression for the exponential generating function for the number of cycles of only *odd* lengths.

(b) What is the exponential generating function for the number of permutations, all of whose cycles have odd lengths?

(c) Let $f(n)$ be the the the number of permutations of length n that contain only cycles of odd lengths, and only contain an even number of cycles. What is the exponential generating function for $f(n)$? *Hint:* You can get only the even terms from $\exp(x)$ as $(1/2)(\exp(x) - \exp(-x))$, also known as the hyperbolic cosine $\cosh(x)$.

(d) Give a relatively simple expression involving binomial coefficients, factorials, and exponentials for $f(n)$

(e) Based on the formula from the previous part, give an alternative interpretation of these numbers in terms of tossing fair coins. Can you give a bijection between this interpretation and the cycle interpretation (may be harder)?

4. A permutation is said to be *even* if the number of even-length cycles it contains is even (it may contain any number of cycles of odd lengths).

(a) What is the exponential generating function for the even-length cycles?

(b) As in the previous question, use part (a) to get the exponential generating function for the number of permutations of length n

that consist entirely of even cycles, and contain an even number of even cycles.

(c) Combine 4(b) and 3(b) using one of the rules for combining exponential generating functions to get the exponential generating function for the number of even permutations. Give a very simple closed-form expression for the number of even permutations of length n. What do you notice? What do you wonder?

Readings on inclusion–exclusion before class Tuesday Oct 10

- van Lint & Wilson Chapter 10, through and including Example 10.3. (Note: includes derangements via inclusion–exclusion). (4 pages)

- Generatingfunctionology Section 4.2 for a generating function view of inclusion–exclusion, at least through Example 2 (6 pages).

- Generatingfunctionology p. 113 for average number of fixed points of a permutation via inclusion–exclusion (1 page)

Resources for these exercises

- Generatingfunctionology p. 45 Example 4 for derangements via exponential generating functions.

- Generatingfunctionology Section 4.1 talks about using generating functions to find averages (relevant to Question 2).

- Generatingfunctionology pp. 40–42 for rules for composing exponential generating functions.

- Generatingfunctionology pp. 73–78 for a view of labeled structures via "cards", "decks", and "hands", and "exponential families". Example 2 is cycles and permutations.

- Problem 3 is covered in Generatingfunctionology Section 3.7.

- Flajolet & Sedgewick Chapter II for labeled structures and operations on exponential generating functions. Especially pp. 102–103 for sequences, sets, and cycles, and Section II.4 for permutations.